Kuwait University | **Math 102** | **July 23, 2011**
Department of Mathematics | **Second Examination** | **Duration : 90 minutes**

The use of calculators of all kinds is not allowed. All communication devices including Mobile phones should be switched off. Answer all of the following questions.

1. Evalute the following integrals. **(3 pts each)**

(a) $\int \tan^5 x dx.$

(b) $\int \cos(\ln x) dx.$

(c) $\int \frac{7x^2 - 4x - 17}{x^3 + 2x^2 + 17x} dx.$

(d) $\int \frac{x^6 + 2}{x^3\sqrt{x^2 - 1}} dx$

(e) $\int \frac{\cos x}{\sin x + \cos x + 1} dx.$

(f) $\int \frac{1}{x\sqrt{1 + \sqrt{x}}} dx.$

2. Determine if the improper integral $\int_0^\infty \frac{dx}{e^{2x} + 9}$ is convergent or divergent. Find its value if it is convergent.

(4 pts)

3. Suppose that f is continuously differentiable on $[0, 1]$ with $f(1) = -3$ and $\int_0^1 x\sqrt{1 - f(x)} dx = 5$. Find

$\int_0^1 \frac{x^2 f'(x)}{\sqrt{1 - f(x)}} dx.$ **(3 pts)**

Solution Key; Second Examination Math 102; Summer Session of 2010/2011

1. (a) $\tan^5 x = (\cos x)^{-5}\sin^5 x = \left((\cos x)^{-5} - 2(\cos x)^{-3} + (\cos x)^{-1}\right)\sin x$

$$\int \tan^5 x dx \overset{u=\cos x}{=} -\int \left(u^{-5} - 2u^{-3} + u^{-1}\right) du = \frac{1}{4u^4} - \frac{1}{u^2} - \ln|u| + C = \frac{1}{4\cos^4 x} - \frac{1}{\cos^2 x} - \ln|\cos x| + C$$

(b) $u = \cos(\ln x) \qquad dv = 1\cdot dx \Rightarrow du = -\frac{\sin(\ln x)}{x}dx \qquad v = x$

$$\begin{aligned}
\int \cos(\ln x)dx &= \int udv = uv - \int vdu = x\cos(\ln x) + \int \sin(\ln x)dx\\
\int \sin(\ln x)dx &= \int wdz = wz - \int zdw \qquad w = \sin(\ln x) \qquad ; dz = dx \Rightarrow dw = \frac{\cos(\ln x)}{x}dx \qquad z = x\\
&= x\sin(\ln x) - \int \cos(\ln x)dx\\
\int \cos(\ln x)dx &= \tfrac{1}{2}x\left(\cos(\ln x) + \sin(\ln x)\right)
\end{aligned}$$

(c) The rational function $\frac{P(x)}{Q(x)} = \frac{7x^2-4x-17}{x(x^2+2x+17)}$ is proper since $\deg P < \deg Q$; $Q(x)$ has been factored into a product of a linear factor: x (repeated once), and an irreducible quadratic factor: $x^2+2x+17$ (repeated once); the Partial Fraction Decomposition has the form $\frac{P(x)}{Q(x)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+17}$.

$$\begin{aligned}
7x^2 - 4x - 17 &= (A+B)x^2 + (2A+C)x + 17A \Rightarrow A = -1;\ B = 7+1 = 8;\ C = -4+2 = -2\\
\frac{P(x)}{Q(x)} &= -\frac{1}{x} + \frac{8x-2}{x^2+2x+17} = -\frac{1}{x} + \frac{8x-2}{(x+1)^2+16} = -\frac{1}{x} + \frac{8(x+1)-10}{(x+1)^2+16}
\end{aligned}$$

all manipulations after the second equal sign are for integration purpose

$$\int \frac{P(x)}{Q(x)}dx = -\ln|x| + 4\ln\left(x^2+2x+17\right) - \frac{5}{2}\tan^{-1}\left(\frac{1}{4}x + \frac{1}{4}\right) + C$$

(d) Use the substitution: $x = \sec\theta$; $dx = \tan\theta\sec\theta d\theta$

$$\begin{aligned}
\int \frac{x^6+2}{x^3\sqrt{x^2-1}}dx &= \int \frac{\sec^6\theta+2}{\sec^3\theta\tan\theta}\tan\theta\sec\theta d\theta = \int \frac{\sec^6\theta+2}{\sec^2\theta}d\theta = \int\left(\sec^4\theta + 2\cos^2\theta\right)d\theta\\
&= \int\left(\left(1+\tan^2\theta\right)\sec^2\theta + 1 + \cos 2\theta\right)d\theta = \tan\theta + \frac{1}{3}\tan^3\theta + \theta + \frac{\sin 2\theta}{2} + C\\
&= \frac{1}{3}\left(x^2+2\right)\sqrt{x^2-1} + \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C
\end{aligned}$$

(e) Use the substitution $u = \tan\frac{x}{2}$, $-\pi < x < \pi$; so that $du = \frac{2}{1+u^2}du$; $\sin x = \frac{2u}{1+u^2}$; $\cos x = \frac{1-u^2}{1+u^2}$

$$\begin{aligned}
\int \frac{\cos x}{\sin x + \cos x + 1}dx &= \int \frac{\frac{1-u^2}{1+u^2}}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2} + 1}\frac{2du}{1+u^2} = \int \frac{1-u^2}{(u+1)(1+u^2)}du = \int \frac{1-u}{1+u^2}du = \tan^{-1}u - \frac{1}{2}\ln(1+u^2) + C\\
&= \tan^{-1}\tan\frac{x}{2} - \frac{1}{2}\ln(1+\tan^2\frac{x}{2}) + C = \frac{x}{2} + \ln(\cos\frac{x}{2}) + C
\end{aligned}$$

(f) Use the substitution $x = \tan^4\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then, $\sqrt{1+\sqrt{x}} = \sec\theta$ and $dx = 4\tan^3\theta\sec^2\theta d\theta$

$$\begin{aligned}
\int \frac{1}{x\sqrt{1+\sqrt{x}}}dx &= \int \frac{4\tan^3\theta\sec^2\theta}{\tan^4\theta\sec\theta}d\theta = 4\int \frac{\sec\theta}{\tan\theta}d\theta = 4\int \csc\theta d\theta = 4\ln|\csc\theta - \cot\theta| + C\\
&= 4\ln\left|\frac{\sqrt{1+\sqrt{x}}}{\sqrt[4]{x}} - \frac{1}{\sqrt[4]{x}}\right| + C = 4\ln\left|\frac{\sqrt[4]{x}}{\sqrt{1+\sqrt{x}}+1}\right| + C = \ln x - 4\ln(1 + \sqrt{1+\sqrt{x}}) + C
\end{aligned}$$

2. $\int \frac{dx}{e^{2x}+9} = \int \frac{e^{-2x}}{1+9e^{-2x}}dx \overset{u=e^{-x}}{=} -\int \frac{u}{1+9u^2}du = -\frac{1}{18}\ln(1+9u^2) = -\frac{1}{18}\ln(1+9e^{-2x})$

$$\int_0^t \frac{dx}{e^{2x}+9} = -\frac{1}{18}\left[\ln(1+9e^{-2t}) - \ln 10\right] \to \frac{1}{18}\ln 10 \text{ as } t\to\infty \Rightarrow \int_0^\infty \frac{dx}{e^{2x}+9} = \frac{1}{18}\ln 10$$

3. Use integration by parts: $u = x^2$, and $dv = \frac{f'(x)}{\sqrt{1-f(x)}}dx$, then, $du = 2xdx$, $v = -2\sqrt{1-f(x)}$, and

$$\begin{aligned}
\int_0^1 \frac{x^2 f'(x)}{\sqrt{1-f(x)}}dx &= \int_0^1 udv = uv|_0^1 - \int_0^1 vdu = -2\,x^2\sqrt{1-f(x)}\Big|_0^1 + 4\int_0^1 x\sqrt{1-f(x)}dx\\
&= -2\sqrt{1-f(1)} + 4\int_0^1 x\sqrt{1-f(x)}dx = -2\sqrt{1-(-3)} + 4(5) = -4 + 20 = 16
\end{aligned}$$